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Busy Beaver Sets and the Degrees of Unsolvability

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§1 Introduction

In this paper we show how some of the finite injury and infinite injury priority arguments can be simplified and in some cases eliminated altogether. This simplification and/or elimination is achieved by making explicit use of the primitive notions of axiomatic computational complexity theory. Phrases such as "perform n steps in the enumeration of W_i" certainly bear witness to the fact that many of these complexity notions have been used implicitly from the early days of recursive function theory. However, other complexity notions such as that of an "honest" function are not so apparent, neither explicitly nor implicitly. Accordingly, one of the main factors in our simplification of these diagonalization arguments is the replacement of the characteristic function χ_A of a set A by the function ν_A, which is the next-element function of the set A. Another important factor is the use of busy beaver sets (see [3]) to provide the basis for the required diagonalizations thereby permitting rather simple and explicit descriptions of the sets constructed.

In preparation for the results which follow we devote the remainder of this section to the requisite definitions and notions as well as some preliminary lemmas. A more comprehensive

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discussion of many of the notions in this section can be found in [3]. Since we will be dealing extensively with relative computations most of our notions here have been correspondingly relativized.

An *acceptable gödel numbering* (see Rogers [8]) is a sequence of functions $\{\phi_i^X\}$ such that

(ϕ1) for every set X of natural numbers, every X- computable partial function is contained in the list $\{\phi_i^X\}$

(ϕ2) there exists a "universal program" u such that for every set X of natural numbers,

$$(\forall i)(\forall p)[\phi_u^X(i,p) = \phi_i^X(p)],$$

(ϕ3) there exists a total computable function S (the S-m-n function) such that for every set X of natural numbers

$$(\forall i)(\forall j)(\forall p)[\phi_{S(i,j)}^X(p) = \phi_i^X(j,p)]$$

We will often say that i is a program for ϕ_i^X, and $\phi_i^X(p)\downarrow$ will denote the fact that $\phi_i^X(p)$ is defined, i.e., halts and $\phi_i^X(p)\uparrow$ will denote the fact that $\phi_i^X(p)$ is undefined. We will abbreviate $\phi_i^\emptyset$ by ϕ_i and use W_i and dom ϕ_i to denote the domain of ϕ_i, i.e.

$$W_i = \{p \mid \phi_i(p)\downarrow\}.$$

Hereafter, X will always denote a set of natural numbers and we will use the following abbreviations

$$X|s = \{p \mid p \in X \text{ and } p \le s\}$$

$$X|(p,q) = \{r \mid r \in X \text{ and } p < r < q\}$$

$$X|(p,q] = \{r \mid r \in X \text{ and } p < r \le q\}$$

A *program transformation* is any total computable function σ of the form $\sigma(i) = S(k,i)$, or $\sigma(i,j) = S(S(k,i),j)$ etc., for some program k.

A *computational complexity measure* $\{\Phi_i^X\}$ for $\{\phi_i^X\}$ (see Blum [1], Lynch [6]) is a sequence of functions such that

(Φ1) for every set X of natural numbers,
dom Φ_i^X = dom ϕ_i^X

(Φ2) for every set X of natural numbers, $\Phi_i^X(p) = q$
is a predicate effectively decidable from
i,p,q, and X.

As before we abbreviate $\Phi_i^\emptyset$ by Φ_i.

A *program size measure* (see Blum [2]) is any total computable function μ such that

(μ1) $\{i \mid \mu(i) = n\}$ is a finite set

(μ2) there exists a total computable
function κ, such that
$\kappa(n)$ = cardinality of $\{i \mid \mu(i) = n\}$.

A *measured universal programming system* (m.u.p.s.) is a system $\Omega = \langle\{\phi_i^X\},\mu,\{\Phi_i^X\}\rangle$ where $\{\phi_i^X\}$ is an acceptable gödel numbering, μ is a program size measure and $\{\Phi_i^X\}$ is a computational complexity measure for $\{\phi_i^X\}$, such that

(Ω1) $(\forall i)(\exists c)(\forall j)[\mu(S(i,j)) = \mu(j)+c]$

(Ω2) $(\forall i)[0 < \mu(i) \le \log_2 i]$

(Ω3) $(\forall i)(\forall p)(\forall x)[\Phi_i^X(p) \ge \max\{\phi_i^X(p),\mu(i),p\}]$.

$$(\Omega 4)\quad (\forall X)(\forall i)(\forall p)(\forall s \geq \Phi_i^{X}(p))$$
$$[\phi_i^{X|s}(p) = \phi_i^{X}(p) \text{ and }$$
$$\Phi_i^{X|s}(p) = \Phi_i^{X}(p)]$$

$$(\Omega 5)\quad (\forall X)(\forall i)(\forall p)(\forall s)$$
$$[\Phi_i^{X|s} \leq s \Rightarrow$$
$$\phi_i^{X|s}(p) = \phi_i^{X}(p) \text{ and } \Phi_i^{X|s}(p) = \Phi_i^{X}(p)]$$

Conditions ($\Omega 4$) and ($\Omega 5$) insure that the computational complexity measure takes into account the members of X "used" in a computation, which is an essential notion of relative computations (see Shoenfield [10]).

From ($\Omega 1$) it follows that for any program transformation σ of the form $\sigma(i) = S(k,i)$,

$$(+)\ (\exists c)(\forall i)[\mu(\sigma(i)) = \mu(i)+c].$$

The following lemma (see van Emde Boas [5]) will be very useful in subsequent sections.

Extended Inefficiency Lemma: There exists a program transformation σ such that

$$(\forall i)(\forall j)(\forall k)[(i)\quad W_{\sigma(i,j,k)} = W_i \cap W_j$$
$$(ii)\quad \mu(\sigma(i,j,k)) > k$$
$$(iii)\quad (\forall p \in W_{\sigma(i,j,k)})[\phi_{\sigma(i,j,k)}(p) = \phi_i(p) \text{ and}$$
$$\Phi_{\sigma(i,j,k)}(p) > \Phi_j(p)]].$$

For any set X of natural numbers the next element function ν_X of X is defined by,

$$\nu_X(p) = \min\{q \mid q > p \text{ and } q \in X\}.$$

We use $\deg X$ to denote the degree of unsolvability of the set of natural numbers X.

Lemma 1.1: $\deg X \leq \deg Y \Leftrightarrow (\exists i)[\phi_i^Y = \nu_X]$.

Proof: The lemma follows from the fact that $\deg X \leq Y \Leftrightarrow (\exists i)[\phi_i^Y = \chi_X]$, and the equations

$$\nu_X(p) = \min\{q \mid q > p \text{ and } \chi_X(q) = 1\}$$

$$\chi_X(p) = \begin{cases} 1, \text{ if } p > 0 \text{ and } \nu_X(p-1) = p, \\ \quad \text{or if } p = 0 \text{ and } p \in X \\ 0, \text{ otherwise} \end{cases}$$

Let $\Omega = \langle\{\phi_i\}, \mu, \{\Phi_i\}\rangle$ be some m.u.p.s.

Let λ be the program transformation such that

$$(\forall i)(\forall j)(\forall p)[\phi_{\lambda(i,j)}(p) = \phi_i(\Phi_j(p)) \text{ and } \Phi_{\lambda(i,j)}(p) > \phi_{\lambda(i,j)}(p)$$

Define the total recursive function f as follows:

$$f(0) = 0$$

$$f(1) = \mu(\min\{i \mid (\forall p)[\phi_i(p) = p]\})$$

$$f(n+1) = \max\{\mu(\lambda(i,j)) \mid \mu(i) \leq n \text{ and } \mu(j) \leq f(n)\}.$$

Then define the program size measure μ° by

$$\mu^\circ(i) = n \Leftrightarrow f(n-1) < \mu(i) \leq f(n).$$

Clearly, $\Omega^\circ = \langle\{\phi_i\}, \mu^\circ, \{\Phi_i\}\rangle$ is a m.u.p.s.

Next define the following functions

$$b^\circ(n) = \max\{\Phi_i(1) \mid \phi_i(1)\downarrow \text{ and } \mu^\circ(i) \le n\}$$

$$g^\circ(p) = \begin{cases} \min\{\mu^\circ(i) \mid \Phi_i(1) = p \text{ and } \mu^\circ(i) \le p\}, \\ \quad \text{if such an } i \text{ exists} \\ p, \text{ otherwise} \end{cases}$$

$$b^\circ(n,s) = \begin{cases} \max\{\Phi_i(1) \mid \Phi_i(1) \le s \text{ and } \mu^\circ(i) \le n\}, \\ \quad \text{if such an } i \text{ exists} \\ s, \text{ otherwise} \end{cases}$$

and the sets

$$B^\circ = \{b^\circ(n)\}$$

$$B^\circ_s = \{b^\circ(n,s) \mid n \le s\}.$$

$A^\circ = \overline{B}^\circ$, the complement of B°.

Clearly, the functions g° and $b^\circ(n,s)$ and the sets B°_s are recursive. The following lemma summarizes the most important properties of the busy beaver set B°.

Lemma 1.2

(a) $(\forall i)(\forall n \ge i)[\phi_i(b^\circ(n))\downarrow \Rightarrow \phi_i(b^\circ(n)) < b^\circ(n+1)]$.

(b) b° is not recursive.

(c) $(\forall n)[b^\circ(n) < b^\circ(n+1)]$.

(d) $(\forall n)(\forall s)[b^\circ(n,s) \le b^\circ(n+1,s)$ and $b^\circ(n,s) \le b^\circ(n,s+1)]$.

(e) $(\forall n)(\forall s \ge b^\circ(n))[b^\circ(n,s) = b^\circ(n)]$.

(f) $(\forall n)[g^\circ(b^\circ(n)) = n]$.

(g) $(\forall p)[b^\circ(g^\circ(p)) \ge p]$.

(h) $(\forall p)[p \in B^\circ \Leftrightarrow b^\circ(g^\circ(p)) = p]$.

(i) A° is recursively enumerable.

(j) $\deg B^\circ = 0' = \deg A^\circ$.

Proof: (a): Suppose $n \geq i \geq \mu(i)$ and let j be such that $\mu^{\circ}(j) = n$ and $\Phi_j(1) = b^{\circ}(n)$. Then $\mu(j) \leq f(n)$ so that $\mu(\lambda(i,j)) \leq f(n+1)$ and hence $\mu^{\circ}(\lambda(i,j)) \leq n+1$. If $\phi_i(b^{\circ}(n))\downarrow$ then $\phi_i(b^{\circ}(n)) = \phi_{\lambda(i,j)}(1) < \Phi_{\lambda(i,j)}(1) \leq b^{\circ}(n+1)$.

(b): Immediate corollary of part (a).

(c): Follows from (a) and the definition of $f(1)$.

(d): Obvious.

(e): Obvious.

(f): Follows from (c) and the definition of g°.

(g): Obvious.

(h): $(\Rightarrow)$

If $p \in B^{\circ}$ then $p = b^{\circ}(m)$ for some m and by part (f) $g^{\circ}(p) = m$ so that $p = b^{\circ}(g^{\circ}(p))$.

$(\Leftarrow)$ Obvious.

(i): $p \in A^{\circ} \Leftrightarrow b^{\circ}(g^{\circ}(p)) > p \Leftrightarrow (\exists q > p)[g^{\circ}(q) = g^{\circ}(p)]$.

Thus the partial computable function defined by

$$\phi(p) = \min\{q \mid q > p \text{ and } g^{\circ}(q) = g^{\circ}(p)\}$$

is such that $\operatorname{dom} \phi = A^{\circ}$.

(j): It suffices to show that the Halting Problem

$$H = \{i \mid \phi_i(i)\downarrow\}$$

is computable from B°. Let γ be a program transformation such that $\phi^{X}_{\gamma(i)}(p,q) = \phi^{X}_{i}(p)$.

$$\text{Then } \phi_i(i)\downarrow \Leftrightarrow \phi_{\gamma(i)}(i,1)\downarrow$$
$$\Leftrightarrow \phi_{S(\gamma(i),i)}(1)\downarrow$$
$$\Leftrightarrow \phi_{S(\gamma(i),i)}(1) \leq b^\circ(\mu^\circ(S(\gamma(i),i)))$$ ⬢

We can also demonstrate the recursive enumerability of A° by exhibiting an effective procedure for enumerating the members of A°. Because of its relevance to subsequent constructions we now give such an enumeration procedure. The procedure uses infinitely many markers $\boxed{m}$ which will be placed at certain times next to positive integers which have been arranged in a list. Marker $\boxed{m}$ corresponds to the set of computations $\{\phi_i(1) | \mu^\circ(i) \leq m\}$ and its final resting place will be adjacent to $b^\circ(m)$. The construction proceeds in stages and at stage s the position of marker $\boxed{m}$ for $m \leq s$ will be adjacent to $b^\circ(m,s)$.

Stage s:

(1) Place marker $\boxed{s}$ next to integer s.

(2) Let $m = g^\circ(s)$ and move all markers $\boxed{m}$, $\boxed{m+1}$, ..., $\boxed{s}$ next to integer s. Enumerate all those integers $\leq s$ which are not adjacent to any markers. ⬢

It is clear from this procedure that B°_s consists of all integers $\leq s$ which are adjacent to some marker. Also, we have

Lemma 1.3: (a) $(\forall n)[B^\circ | b^\circ(n) = B^\circ_{b^\circ(n)}]$.

(b) $(\forall s)[B^\circ | s = B^\circ_s \Leftrightarrow s \in B^\circ]$.

§2 Post's Problem

In this section we construct a recursively enumerable set which is neither recursive nor complete thereby providing a solution to the well known problem posed by Emil Post in [7]. Our solution unlike previous solutions is the result of a straight forward diagonalization which does not involve the finite injury priority method. In order to provide suitable motivation for our construction and to contrast it with the finite injury priority method we first present a summary of the important points of this method. We refer the reader to Soare [11] for a more complete description of the finite injury priority method.

Suppose that C is a complete recursively enumerable set. To solve Post's problem we must construct a recursively enumerable set A such that for each program i the following conditions are satisfied:

$$(N_i) \quad \chi_C \neq \phi_i^A$$

$$(P_i) \quad \chi_A \neq \phi_i.$$

The condition (N_i) is called a negative requirement since in the priority construction the satisfaction of this condition may require that certain integers be restrained from being put into the set A. The priority construction proceeds in stages, and by A_s we denote those integers $\leq s$ which have been put into A by stage s and by $\overline{A}_s$ we denote all those integers $\leq s$ which have not been placed into A by stage s. Similarly by C_s ($W_{j,s}$) we denote those integers $\leq s$ which have been put into C (W_j) by stage s of some enumeration procedure for C (W_j). Three cases must be distinguished in the satisfaction of (N_i) by some integer p.

Case 1 : At some stage s, $\chi_{C_s}(p) = 1$ and $\phi_i^{A_s}(p) = 0$.

In this case the computation $\phi_i^{A_s}(p)$ is preserved by restraining from A all integers belonging to $\overline{A}_s$ (i.e., by setting $A|s = A_s$). Since C is recursively enumerable, $\chi_{C_s}(p) = 1 \Rightarrow \chi_C(p) = 1$ and the disagreement between χ_C and ϕ_i^A is established.

Case 2 : At some stage s, $\chi_{C_s}(p) = 0$ and $\phi_i^{A_s}(p) = 1$.

Again the computation $\phi_i^{A_s}(p)$ is preserved by setting $A|s = A_s$ but success is no longer assured since it is possible that for some stage $t > s$ $\chi_{C_t}(p) = 1$. This uncertainty represents a major difficulty in describing the set A which is constructed.

Case 3 : For all but finitely many stages s,

$$\phi_i^{A_s}(p)\uparrow.$$

In this case nothing is required by the construction since $\phi_i^A(p)\uparrow$ also in this case. However, that this case applies can only be verified when one has reached eternity, and the eternal uncertainty also poses a major difficulty in the description of A.

Since C is not recursive, for infinitely many p one of these three cases is guaranteed to hold.

The condition (P_i) is called a positive requirement, since in its more commonly phrased equivalent

$$W_i \text{ infinite } \Rightarrow W_i \cap A \neq \emptyset,$$

(which itself is equivalent to: W_i infinite $\Rightarrow W_i \neq \overline{A}$) its satisfaction may entail putting integers into the set A. The difficulty in arriving at a description for A posed by the condition (P_i) is the uncertainty of whether or not W_i is infinite.

It is possible for these two conditions to be in direct conflict with one another. For example, (N_i) may require that a particular integer p be restrained from A (because for some stage $s \geq p$ and some integer $q \leq s$, $\chi_{C_s}(q) \neq \phi_i^{A_s}(q)$ and $p \in \overline{A}_s$) and (P_j) may require that the same integer p be put into A (because for some stage $t > s$, $p \in W_{j,t}$). This conflict is resolved by giving (P_j) precedence over (N_i) if and only if $j < i$. More precisely, when the above conflict first arises at stage t, p is placed into A_t if $j < i$, and is restrained from A_t otherwise. If $j < i$ and p is placed into A_t then the computation $\phi_i^{A_t}(q)$ may no longer be valid and therefore a new witness for (N_i) may have to be found, and if $j \geq i$ then a new witness for (P_j) will have to be found. Thus the resolution of these conflicts, which is the heart of the finite injury priority method, requires restarting certain diagonalizations, which in turn makes the description of the set constructed extremely complex.

Suppressing the adversary relationship between the conditions (N_i) and (P_j) we see that the essence of the diagonalization is the discovery of witnesses n_i and p_i to the conditions (N_i) and (P_i) respectively. The priority placed on the conditions dictates that $n_i < p_i$ and $p_i < n_{i+1}$. From our discussion above concerning the preservation of computations it is clear that p_i should be larger than the largest integer used in the computation $\phi_i^A(n_i)$. Translating this condition into complexity terms via (Ω4) we obtain the condition $\Phi_i^A(n_i) < p_i$, and via (Ω5) the condition $\Phi_i^{A|p_i}(n_i) < p_i$. To remove the difficulty caused by the uncertainty of Case 2 we replace χ_C with ν_C in the diagonalization, and to remove the difficulty caused by the uncertainty of Case 3 we choose A° as the complete recursively enumerable set C. The use of A° in our construction has the added property that the positive requirements (P_i) are automatically satisfied. Consequently no conflicts between negative and positive requirements can occur. The witness n_i is in fact equal to $b^\circ(i)$ and a witness to (P_i) will be any

$b°(n)$ for which $n > i$ and $\phi_{\tau(n)}(b°(n))\uparrow$, where τ is a program transformation described below.

We will find it convenient to diagonalize with respect to the complements of the recursively enumerable sets A and $A°$. Denoting the complement of our desired recursively enumerable set A by B we will construct A so that $\phi_i^B \neq \nu_{B°}$ and $\phi_i \neq \nu_B$ for each program i. The desired properties of A then follow from the fact that a set and its complement have the same degree of unsolvability. We proceed now with the formal details. We first define the following recursive functions and sets by induction:

$$\phi_{\tau(i)}(p) = \min\{s \mid s > p \text{ and } \Phi_i^{B_s}(p) \leq s\}$$

$$h_s(q) = \begin{cases} \min\{p \mid p \varepsilon B_s^° \text{ and } p > \phi_{\tau(g°(q))}(q)\}, \\ \qquad \text{if } \phi_{\tau(g°(q))}(q) < s \\ \min\{p \mid p \varepsilon B_s^° \text{ and } p > q\}, \text{ otherwise} \end{cases}$$

$$\sigma_{1,s} = b°(1,s)$$

$$\sigma_{k+1,s} = \begin{cases} h_s(\sigma_{k,s}), \text{ if } \sigma_{k,s} < s \\ s, \text{ otherwise} \end{cases}$$

$$k_s = \min\{k \mid \sigma_{k,s} = s\}$$

$$B_s = \{\sigma_{1,s}, \ldots, \sigma_{k_s,s}\}$$

$$B_s^1 = B_s - \{s\}$$

$$\sigma_s = \begin{cases} \min\{\sigma \mid \sigma \in B_s \text{ and } \phi_{\tau(g^\circ(\sigma))}(\sigma) = s\}, \text{ if there} \\ \quad \text{exists such a } \sigma \\ s, \text{ otherwise} \end{cases}$$

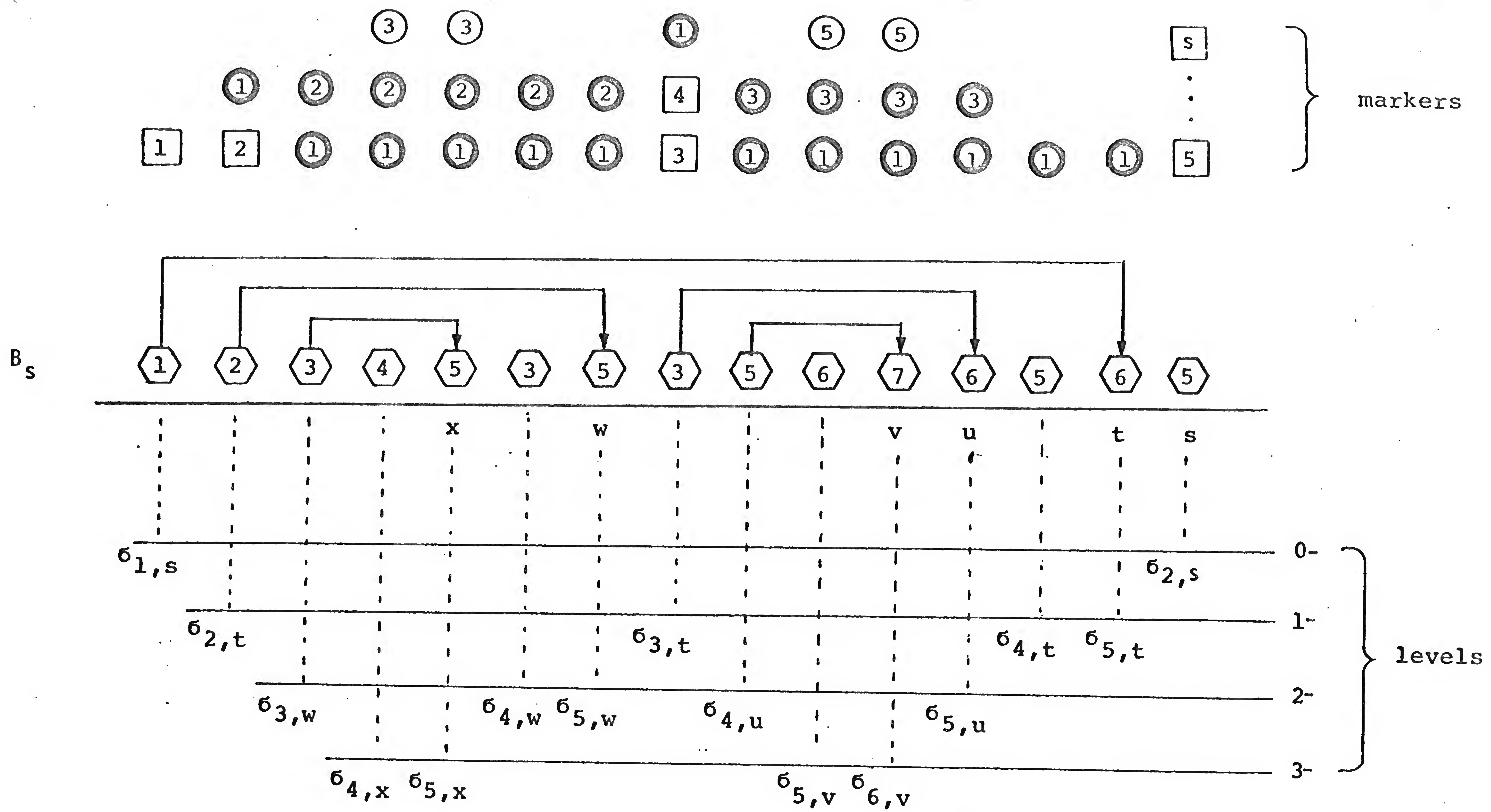
$$B_s = B_s \cup \bigcup_{\sigma \in B_s^1} B(\sigma, h_s(\sigma))$$

$$B(p,q) = \begin{cases} B_s \mid (p,s], \text{ if } \phi_{\tau(g^\circ(p))}(p) = s < q \\ \emptyset, \text{ otherwise} \end{cases}$$

The above functions and sets are well defined since to decide whether $\phi_{\tau(g^\circ(p))}(p) < q$ requires knowledge of B_s only for $s < q$, and B_s is defined in terms of $B(p,q)$ for $p,q \leq s$. In view of (Φ2) the predicate $\phi_{\tau(i)}(n) = s$ is effectively decidable in i,n,s and B_s. It is clear from its definition that B_s has a recursive structure. This structure is depicted in figure 1 below. In order to more easily deal with this nested structure we make the following definitions: We say that p is a *level* 0 *member* of B_s if and only if $p \in B_s$ and $p \leq \sigma_s$. Moreover, if $p \in B_s^1$ we say that p is an *interior level* 0 *member* of B_s. We say that p is a *level* r *member* of B_s if and only if there exists an interior level $r-1$ member σ of B_s such that

$$\sigma < p < \phi_{\tau(g^\circ(\sigma))}(\sigma) = t \leq s \quad \text{and}$$

$$p \in B_t.$$


markers
levels
B_s
3 3 1 5 5 s
1 2 2 2 2 2 4 3 3 3 3
1 2 1 1 1 1 1 3 1 1 1 1 1 1 5
1 2 3 4 5 3 5 3 5 6 7 6 5 6 5
x w v u t s
0- 1- 2- 3-
$\sigma_{1,s}$ $\sigma_{2,s}$
$\sigma_{2,t}$ $\sigma_{3,t}$ $\sigma_{4,t}$ $\sigma_{5,t}$
$\sigma_{3,w}$ $\sigma_{4,w}$ $\sigma_{5,w}$ $\sigma_{4,u}$ $\sigma_{5,u}$
$\sigma_{4,x}$ $\sigma_{5,x}$ $\sigma_{5,v}$ $\sigma_{6,v}$

Legend: (1) ⟨i⟩ next to integer p indicates that $\boxed{i}$ was adjacent to p at stage p

(2) ⟨i⟩ ⟨j⟩ indicates that $\phi_{\tau(i)}(b^\circ(i,p)) = p$.
p

(3) ● - active phantom marker; ○ - inactive phantom marker; □ - real marker

FIGURE 1

Moreover if $p \in \mathcal{B}_t^1$ then we say that p is an *interior level* r *member* of B_s. From the definition of B_s we have that

(2.1) $p \in B_s \Leftrightarrow$ there exists an $r \geq 0$ such that p is a level r member of B_s.

Let

$$m_\circ = \min\{m \mid \mu^\circ(\tau(m)) \leq m\} \quad \text{and}$$

$$n_\circ = \min\{n \mid b^\circ(n) > \max\{\phi_{\tau(m)}(b^\circ(m)) \mid \phi_{\tau(m)}(b^\circ(m))\downarrow \text{ and } m \leq m_\circ\}$$

The existence of $m_\circ$ follows from (+). The set B is defined in segments as follows:

$$B|b^\circ(n) = B_{b^\circ(n)}$$

for $n \geq n_\circ$

$$B|(b^\circ(n), b^\circ(n+1)] = \mathcal{B}(b^\circ(n), b^\circ(n+1)) \cup \{b^\circ(n+1)\}.$$

Clearly $B^\circ \subseteq B$. We note that if $\phi_{\tau(n)}(b^\circ(n)) \geq b^\circ(n+1)$ for $n > n_\circ$ the

$$B|(b^\circ(n), b^\circ(n+1)] = \{b^\circ(n+1)\} \quad \text{and} \quad \nu_B(b^\circ(n)) = b^\circ(n+1).$$

We will show that the set B has the desired properties through a series of lemmas. Our first lemma follows immediately from the definition of B and the fact that $\mathcal{B}$ is a recursive set function.

Lemma 2.1: $\deg B \leq \deg B^\circ$.

Lemma 2.2: (a) $(\forall n \geq n_\circ)(\forall t \geq b^\circ(n)[B_{b^\circ(n)} = B_t|b^\circ(n)]$.

(b) $(\forall n \geq n_\circ)[B|b^\circ(n) = B_{b^\circ(n)}]$.

(c) $(\forall n \geq n_\circ)[\phi_{\tau(n)}(b^\circ(n)) = s \Rightarrow s < b^\circ(n+1)$ and $B_s = B|s]$.

Proof: (a): Let $n > n_0$, $s = b^\circ(n)$ and $t > s$. From Lemma 1.2 it clearly follows that

$$(\forall m \le n)[b^\circ(m,s) = b^\circ(m) = b^\circ(m,t)].$$

Also, since $n \ge n_0$, for all $m < n$

$$\phi_{\tau(m)}(b^\circ(m))\downarrow \Leftrightarrow \phi_{\tau(m)}(b^\circ(m)) < b^\circ(n) = s$$

and consequently

$$\phi_{\tau(m)}(b^\circ(m,t)) < t \Leftrightarrow \phi_{\tau(m)}(b^\circ(m,s)) < s.$$

Therefore, for all $k \le k_s$, $\sigma_{k,s} = \sigma_{k,t}$ and $\sigma_{k_s,t} = s$.

$$\text{Thus } B_t|s = B_t|s \cup \bigcup_{1 \le k < k_s} \mathcal{B}(\sigma_{k,t}, \sigma_{k+1,t})$$

$$= B_s \cup \bigcup_{\sigma \in B_s^1} \mathcal{B}(\sigma, h_s(\sigma)) = B_s$$

(b): The proof is by induction on n. Clearly for $n = n_0$, $B|b^\circ(n) = B_{b^\circ(n)}$ follows from the definition of B. Suppose that $B|b^\circ(n) = B_{b^\circ(n)}$ for some $n \ge n_0$ and let $s = b^\circ(n+1)$. From part (a) above we have that

$$B_{b^\circ(n+1)}|b^\circ(n) = B_{b^\circ(n)} = B|b^\circ(n).$$

Since $n \ge n_0$ by Lemma 1.2(a) we have that if $\phi_{\tau(n)}(b^\circ(n))\downarrow$ then $\phi_{\tau(n)}(b^\circ(n)) < b^\circ(n+1)$ and hence that $\sigma_{k_s-1,s} = b^\circ(n)$. Therefore,

$$B_{b^\circ(n+1)} = B_{b^\circ(n)} \cup \{\sigma_{k_s,s}\} \cup \mathcal{B}(b^\circ(n), b^\circ(n+1))$$

$$= B|b^\circ(n) \cup \{b^\circ(n+1)\} \cup \mathcal{B}(b^\circ(n), b^\circ(n+1))$$

$$= B|b^\circ(n+1).$$

(c): Let $n > n_\circ$ and let $s = \phi_{\tau(n)}(b^\circ(n))$. From our remarks in the proof of part (b) above it is clear that $s < b^\circ(n+1)$. Since by parts (a) and (b) above $B|b^\circ(n) = B_{b^\circ(n)} = B_s|b^\circ(n)$, it suffices to show that $B|(b^\circ(n),s] = B_s|(b^\circ(n),s]$. Since $n \geq n_\circ$ and $\phi_{\tau(n)}(b^\circ(n)) = s < b^\circ(n+1)$, it follows that

$$B|(b^\circ(n),s] = B(b^\circ(n),b^\circ(n+1))|s = B_s|(b^\circ(n),s]. \quad ⬡$$

Lemma 2.3: deg $B \neq$ deg B°.

Proof: We show that for each program i, $\phi_i^B \neq \nu_{B^\circ}$. Given i choose a $j \geq n_\circ$ via the Extended Inefficiency Lemma such that $\phi_j^X = \phi_i^X$ for every set X. If $\phi_j^B(b^\circ(j))\uparrow$ then $\phi_i^B = \phi_j^B \neq \nu_{B^\circ}$, so we will assume that $\phi_j^B(b^\circ(j))\downarrow$. Let $t \geq \Phi_j^B(b^\circ(j))$ be such that $t = b^\circ(n)$ for some $n > j$. By (Ω4) we have $\Phi_j^{B|t}(b^\circ(j)) = \Phi_j^B(b^\circ(j))$ and by Lemma 2.2(b) we have $B|t = B_t$. Therefore $\Phi_j^{B_t}(b^\circ(j)) = \Phi_j^B(b^\circ(j)) \leq t$ and consequently $\phi_{\tau(j)}(b^\circ(j))\downarrow$.

Let $s = \phi_{\tau(j)}(b^\circ(j))$. Since $j > n_\circ$, from Lemma 2.2(c) it follows that $s < b^\circ(j+1)$ and $B_s = B|s$. Thus

$$(2.2) \quad \Phi_j^{B|s}(b^\circ(j)) = \Phi_j^{B_s}(b^\circ(j)) \leq s$$

from which it follows by (Ω5) that

$$(2.3) \quad \Phi_j^B(b^\circ(j)) = \Phi_j^{B|s}(b^\circ(j)).$$

Combining (2.2) and (2.3) and the fact that $\phi_j^B < \Phi_j^B$ we obtain

$$\phi_j^B(b^\circ(j)) \le s < b^\circ(n+1)$$

but $\nu_{B^\circ}(b^\circ(j)) = b^\circ(j+1)$. Therefore $\nu_{B^\circ} \ne \phi_j^B = \phi_i^B$. ⬡

Lemma 2.4: deg $B \ne 0$.

Proof: Given a program i choose a program $j \ge \max\{n_\circ, i\}$ such that $\phi_j^X(p)\uparrow$ for every set X and every integer p. Clearly $\phi^X_{\tau(j)}(p)\uparrow$ for all X and p also, so that $B|(b^\circ(j), b^\circ(j+1)] = \{b^\circ(j+1)\}$ and $\nu_B(b^\circ(j)) = b^\circ(j+1)$. Since $j \ge \max\{i, n_\circ\}$, if $\phi_i(b^\circ(j))\downarrow$ then $\phi_i(b^\circ(j)) < b^\circ(j+1)$ so that $\phi_i \ne \nu_B$. ⬡

We give a procedure which enumerates the members of a recursively enumerable set A and then show that $\bar{A} = B$. The procedure uses two types of markers: real markers $\boxed{m}$ and phantom markers ⓝ. The real markers are the same as those used in the enumeration procedure for A° given in §1. Indeed, the position of marker $\boxed{m}$ at stage $s \ge m$ is $b^\circ(m,s)$ and its final resting place is $b^\circ(m)$. In contrast to real markers, the phantom markers are never moved or removed and there may be several markers ⓝ for each n. (Recall that there is only one marker $\boxed{m}$ for each m.) Also, every marker has two states: an active state and an inactive state. Real markers are always active, and phantom markers are active when they are first placed next to an integer and may later be deactivated. Once deactivated, however, a phantom marker can never again be active. The set A_s, which is the stage s approximation to A, consists of all integers $\leq s$ which are not adjacent to some active marker. By $\overline{A}_s$ we mean the stage s approximation to $\overline{A}$, i.e., the set of all integers $\leq s$ which are adjacent to some active marker. We now define the enumeration procedure by stages.

Stage s:

1) Place real marker $\boxed{s}$ next to integer s.
2) Let $m = g^\circ(s)$.
   - a) Move all real markers $\boxed{m}$, $\boxed{m+1}$, ..., $\boxed{s}$ next to integer s and
   - b) deactivate all phantom markers $\textcircled{m}$, $\textcircled{m+1}$, ..., $\textcircled{s}$.
3) Let n be the least integer $\leq s$ such that
   - (i) $\phi_{\tau(n)}(b^\circ(n,s)) = s$
   - (ii) $g^\circ(b^\circ(n,s)) = n$, and
   - (iii) $b^\circ(n,s)$ is not adjacent to any active phantom marker.

   If no such integer exists go to stage $s+1$, otherwise place an active phantom marker $\textcircled{n}$ next to each integer between $b^\circ(n,s)+1$ and s which is already adjacent to an active marker (i.e., next to each member of $\overline{A}_s|(b^\circ(n,s),s])$. Go to stage $s+1$. ⬢

We first remark that this procedure can be simplified without affecting the resulting set by removing phantom markers in step (2)(b) instead of deactivating them, thereby eliminating the distinction between active and inactive markers. However, this added detail will simplify the subsequent proof that $\bar{A} = B$. Notice that only during step (2) can new integers be placed into A, and also that while an integer may be adjacent to several phantom markers they must all be distinct. Moreover, a real marker can never be adjacent to an inactive phantom marker.

Lemma 2.5: A is recursively enumerable.

Proof: Clearly, A_s is recursive so it suffices to show that if $p \in A_s$ then $p \in A_t$ for all $t \geq s$. This follows from the fact that at any stage $t > s$ the only integer which can be adjacent to an active marker at state t which was not already adjacent to an active marker at stage t-1 is t, but $t \notin A_{t-1}$. ⬢

Lemma 2.6 (a) $(\forall n \geq n_0)[A_{b^\circ(n)} = A|b^\circ(n)]$.

(b) If $p \in \bar{A}_s$, then p is adjacent to marker ⓜ at stage s if and only if there exists a $t \leq s$ such that

(i) $t = \phi_{\tau(m)}(b^\circ(m,p))$

(ii) $p \in \bar{A}_t$

(iii) $g^\circ(p) > m$

(iv) $b^\circ(m,p) < p \leq t$

(v) $g^\circ(b^\circ(m,p)) = m$

(vi) $b^\circ(m,p) = b^\circ(m,t)$

(vii) $b^\circ(m,p)$ is not adjacent to any phantom markers at stage t

(viii) $(\forall n < m)[\phi_{\tau(n)}(b^\circ(n,t)) \neq t]$.

(c) If $p \in \overline{A}_s$ and p is adjacent to markers $\textcircled{m_1}$ and $\textcircled{m_2}$ at stage s with $m_1 < m_2$ then

(i) $\phi_{\tau(m_1)}(b^\circ(m_1,p)) > \phi_{\tau(m_2)}(b^\circ(m_2,p))$ and,

(ii) $b^\circ(m_2,p)$ is adjacent to marker $\textcircled{m_1}$ at stage s.

(d) If $p \in \overline{A}_s$ and p is adjacent to marker $\textcircled{m_1}$ and $b^\circ(m_1,p)$ is adjacent to marker $\textcircled{m_2}$ at stage s, then p is adjacent to $\textcircled{m_2}$ at stage s.

Proof: (a): Let $n \geq n_0$ and let $s = b^\circ(n)$. By virtue of Lemma 2.5 it suffices to show that if $p \in \overline{A}_s$ then for all $t \geq s$, $p \in \overline{A}_t$. From the enumeration procedure we see that if p is adjacent to an active phantom marker $\textcircled{m}$ at stage s then $p > b^\circ(m,s)$. Since for all $m \leq n$ $b^\circ(m,s) = b^\circ(m)$ at any stage $t > s$, we have $g^\circ(t) > n$ so that none of the real markers $\boxed{m}$ for $m \leq n$ can be moved after stage s and consequently no phantom marker $\textcircled{m}$ adjacent to any integer $\leq s$ can be deactivated after stage s. Hence $A_s = A_t|s$ for all $t \geq s$.

(b): The conditions (i) - (viii) follow directly from the description of the enumeration procedure.

(c): Suppose $p \in \overline{A}_s$ and p is adjacent to markers $\textcircled{m_1}$ and $\textcircled{m_2}$ at stage s with $m_1 < m_2$. Let $t_1 = \phi_{\tau(m_1)}(b^\circ(m_1,p))$ and $t_2 = \phi_{\tau(m_2)}(b^\circ(m_2,p))$. By part (b) we have

$$b^\circ(m_1,t_1) = b^\circ(m_1,p) < p \leq t_1 \leq s \quad \text{and}$$

$$b^\circ(m_2,t_2) = b^\circ(m_2,p) < p < t_2 \leq s,$$

from which it follows that

(2.4) $b^\circ(m_2,t_1) = b^\circ(m_2,p) < t_1$.

Now $b^\circ(m_1,p) < b^\circ(m_2,p)$, since $g^\circ(b^\circ(m_1,p)) = m_1 < m_2 = g^\circ(b^\circ(m_2,p))$ so that by part (b) above and (2.4) we have $b^\circ(m_2,p)$ is is adjacent to marker $\textcircled{m_1}$ at stage t_1. Now if $t_1 \leq t_2$ then $b^\circ(m_2,p)$ would be adjacent to marker $\textcircled{m_1}$ at stage t_2 also, which would contradict the fact that p is adjacent to marker $\textcircled{m_2}$ at stage s.

(d): Suppose $p \in \overline{A}_s$ and p is adjacent to marker $\textcircled{m_1}$ at stage s and $b^\circ(m_1,p)$ is adjacent to marker $\textcircled{m_2}$ at stage s. Let $t_1 = \phi_{\tau(m_1)}(b^\circ(m_1,p))$ and $t_2 = \phi_{\tau(m_2)}(b^\circ(m_2,p))$. By part (b) we have

$$b^\circ(m_1,t_1) = b^\circ(m_1,p) < p \leq t_1 \leq s \quad \text{and,}$$

$$b^\circ(m_2,t_2) = b^\circ(m_2,p) < p \leq t_2 \leq s.$$

Since $b^\circ(m,p)$ cannot be adjacent to marker $\textcircled{m_2}$ at stage t_1, we must have $t_1 < t_2$ and hence

$$b^\circ(m_2,p) < p < t_2.$$

Therefore by part (b), p is adjacent to marker $\textcircled{m_2}$ at stage s. ⬡

Lemma 2.7: p is a level 0 member of $B_s \Leftrightarrow p \in \bar{A}_s$ and p is adjacent to no phantom markers at stage s.

Proof: Since each level 0 member of B_s is a member of B_s° and each member of $\bar{A}_s$ not adjacent to any phantom markers at stage s is also a member of B_s°, it suffices to restrict our attention to members of this set. The proof proceeds by induction on the members of B_s°. Clearly, $b^\circ(1,s) = \sigma_{1,s} \leq \bar{\sigma}_s$, and by virtue of Lemma 2.6(b)(v), $b^\circ(1,s)$ is never adjacent to any phantom markers at stage s.

Now consider $b^\circ(n,s) \leq s$ such that $g^\circ(b^\circ(n,s)) = n$, and assume that for all $m < n$, $b^\circ(m,s)$ is a level 0 member of B_s if and only if $b^\circ(m,s) \in \bar{A}_s$ and $b^\circ(m,s)$ is adjacent to no phantom markers at stage s. Let

$$\sigma = \max\{p \in B_s \mid p < b^\circ(n,s)\}.$$

If $\sigma \geq \bar{\sigma}_s$ then $b^\circ(n,s)$ is not a level 0 member of B_s, and since

$$\bar{\sigma}_s < b^\circ(n,s) \leq \phi_{\tau(g^\circ(\bar{\sigma}_s))}(\bar{\sigma}_s) = s,$$

it follows from Lemma 2.6(b) and the induction hypothesis applied to $\bar{\sigma}_s$ that $b^\circ(n,s)$ is adjacent to marker $\textcircled{g^\circ(\bar{\sigma}_s)}$ at stage s. Suppose $\sigma < \bar{\sigma}_s$. We consider two cases.

Case 1: $\phi_{\tau(g^\circ(\sigma))}(\sigma) < b^\circ(n,s)$ or $\phi_{\tau(g^\circ(\sigma))}(\sigma) > s$

From the definition of h_s we see that

$$b^\circ(n,s) = h_s(\sigma) \in B_s,$$

and since $b^\circ(n,s) < \overline{\sigma}_s$ it follows that $b^\circ(n,s)$ is a level 0 member of B_s. From the induction hypothesis we see that for each $m < n$ if $b^\circ(m,s)$ is not adjacent to any phantom markers at stage s and $\phi_{\tau(m)}(b^\circ(m,s)) \leq s$ then $\phi_{\tau(m)}(b^\circ(m,s)) < b^\circ(n,s)$. Thus $b^\circ(n,s)$ is not adjacent to any phantom markers at stage s.

Case 2: $b^\circ(n,s) < \phi_{\tau(g^\circ(\sigma))}(\sigma) < s$.

From the definition of h_s we see that $h_s(\sigma) > b^\circ(n,s)$ so that $b^\circ(n,s) \notin B_s$ and hence $b^\circ(n,s)$ is not a level 0 member of B_s. Since by the induction hypothesis σ is not adjacent to any phantom markers at stage s, from Lemma 2.6(b) it follows that $b^\circ(n,s)$ is adjacent to $g^\circ(\sigma)$ at stage s. ⬡

Lemma 2.8: $\overline{A} = B$.

Proof: By virtue of Lemma 2.2(b) and Lemma 2.6(a) it suffices to show that for every stage s $\overline{A}_s = B_s$. This is accomplished by showing that p is a level r member of B_s if and only if $p \in \overline{A}_s$ and p is adjacent to exactly r phantom markers at stage s. The proof proceeds by induction on r. The case r = 0 is Lemma 2.7, so we will assume that the above statement is true for some $r \geq 0$.

Suppose that p is a level $r+1$ member of B_s and let σ be the level r member of B_s such that $\sigma < p < \phi_{\tau(g^\circ(\sigma))}(\sigma) = t \leq s$ and $p \in B_t$. In view of the induction hypothesis applied to σ let $\textcircled{m_1}, \ldots, \textcircled{m_r}$ with $m_1 < m_2 < \cdots < m_r < g^\circ(\sigma)$ be the phantom markers adjacent to σ. By Lemma 2.6(b) p is adjacent to marker $\textcircled{g^\circ(\sigma)}$ at stage s and by Lemma 2.6(d) p is also adjacent to $\textcircled{m_1}, \ldots, \textcircled{m_r}$ at stage s. That $\textcircled{m_1}, \ldots, \textcircled{m_r}, \textcircled{g^\circ(\sigma)}$ are the only markers adjacent to p at stage s can be seen as follows. Suppose p is adjacent to marker $\textcircled{m}$ at stage s. If $m < g^\circ(\sigma)$ then by Lemma 2.6(c) $\textcircled{m}$ is adjacent to σ so that $m = m_i$ for some $1 \leq i \leq r$. If $m > g^\circ(\sigma)$ then by Lemma 2.6(c)

$$\sigma < b^\circ(m,p) < p \leq \phi_{\tau(m)}(b^\circ(m,p)) < t$$

so that $p \in \bar{A}_t$ and p is adjacent to the marker $\textcircled{m}$ at stage t and hence by Lemma 2.7 $p \notin B_t$.

Suppose that p is adjacent to markers $\textcircled{m_1}, \ldots, \textcircled{m_{r+1}}$ at stage s. Then by Lemma 2.6(c) $b^\circ(m_{r+1},p)$ is adjacent to markers $\textcircled{m_1}, \ldots, \textcircled{m_r}$ at stage s. By the induction hypothesis $b^\circ(m_{r+1},p)$ is a level r member of B_s. Let

$t = \phi_{\tau(m_{r+1})}(b^\circ(m_{r+1},p))$. By Lemma 2.6(c) for each

$1 \leq i \leq r$, $\Phi_{\tau(m_i)}(b^\circ(m_i,p)) > t \geq p$ so that

$b^\circ(m_{r+1},p) = b^\circ(m_{r+1},t)$ is adjacent to no phantom markers at stage t and hence $p \in B_t$. Therefore p is a level $r+1$ member of B_s. ⬢

Putting all the preceding lemmas together we obtain

Theorem 2.1: (Friedberg-Muchnik) There exists a recursively enumerable set A such that $0 < \deg A < 0'$.

Actually, the set A has an additional property of some interest. From the proof of Lemma 2.3 we see that

$$\phi_i^B(b^\circ(i))\downarrow \;\Rightarrow\; \phi_{\tau(i)}(b^\circ(i))\downarrow.$$

Also if $\phi_{\tau(i)}(b^\circ(i))\downarrow$ then letting $s = \phi_{\tau(i)}(b^\circ(i))$ we see that

$$\Phi_i^{B_s}(b^\circ(i)) \leq s$$

and by Lemma 2.2 that

$$\Phi_i^{B|s}(b^\circ(i)) \leq s$$

and by $(\Omega 5)$ that

$$\Phi_i^B(b^\circ(i)) \leq s$$

so that $\phi_i^B(b^\circ(i))\downarrow$. Therefore we have

$$\phi_i^B(b^\circ(i))\downarrow \;\Leftrightarrow\; \phi_{\tau(i)}(b^\circ(i))\downarrow \;\Leftrightarrow\; \phi_{\tau(i)}(b^\circ(i)) < b^\circ(i+1).$$

Now, let γ be the program transformation defined in §1 so that

$$\phi_i^B(p)\downarrow \;\Leftrightarrow\; (\forall q)[\phi^B_{\gamma(i)}(p,q)\downarrow] \;\Leftrightarrow\; (\exists q)[\phi^B_{\gamma(i)}(p,q)\downarrow]$$

Let $n = S(\gamma(i),i)$. Then

$$\phi_i^B(i)\downarrow \;\Leftrightarrow\; \phi^B_{S(\gamma(i),i)}(b^\circ(n))\downarrow \;\Leftrightarrow\; \phi_{\tau(n)}(b^\circ(n)) < b^\circ(n+1).$$

Therefore, we have $\deg A' = \deg B' = 0'$.

Corollary 2.2: There exists a recursively enumerable set A such that $0 < \deg A < 0'$ and $\deg A' = 0'$.

The rather lengthy proof of Theorem 2.1 above is due for the most part to our concern for describing the set A. If we forego this concern, then a quite simple proof based on the properties of the enumeration procedure for A can be given. In particular, the crucial Lemma 2.2 is an easy consequence of the construction. The enumeration procedure itself can be simplified by removing phantom markers instead of deactivating them. In the subsequent sections we will adopt this approach. While descriptions of the sets constructed will be given, the proofs of the lemmas will be based on the enumeration procedures.

§3. Other Finite Injury Priority Constructions

Not all finite injury priority arguments can be eliminated using the techniques of the preceding sections. Indeed, one of the crucial elements of the construction given in §2 was the use of the set $B°$. More precisely, the property of $B°$ stated in Lemma 1.2(a), which permitted the automatic satisfaction of the positive requirements (which in turn insured zero injuries to the negative requirements), was essential to that construction. But Degtev [4] has shown that any recursively enumerable set whose complement has the property in Lemma 1.2(a) must be complete.

In this section, then, we show how those techniques can be applied to simplify finite injury priority constructions, and while they cannot altogether eliminate injuries to the requirements they can minimize the adversary relationship between the positive and negative requirements. We will show that the class of non-recursive recursively enumerable degrees does not have a minimal element by showing

Theorem 3.1: For every e such that $0 < \deg W_e < 0'$ there exists a recursively enumerable set A such that $0 < \deg A < \deg W_e$.

To prove this result we construct for each e a set B_e such that $\deg B_e = \deg W_e$. As before we have negative and positive requirements to satisfy and using B to denote $\overline{A}$, where A is the desired recursively enumerable set, these requirements take the form

$(N_i)\quad \phi_i^{\,B} \neq \nu_{B_e}$

$(P_i)\quad \phi_i \neq \nu_B.$

The construction can be described along the lines suggested in the beginning of §2 as a consequitive search of B_e for witnesses to the requirements in order according to their priorities, i.e., find a witness n_1 to (N_1) first, then a witness p_1 to (P_1), then a witness n_2 to (N_2) and so on. Before proceeding with the details of the construction we need to consider some preliminary definitions and lemmas.

We assume that some m.u.p.s. $\Omega = \langle\{\phi_i\},\mu,\{\Phi_i\}\rangle$ has been chosen such that $(\forall p)[\phi_0(p)\uparrow]$. Then define

$$b_e(n) = \max\{\Phi_e(p) \mid p \leq n \text{ and } \phi_e(p)\downarrow\}$$

$$b_e(n,s) = \max\{\Phi_e(p) \mid p \leq n \text{ and } \Phi_e(p) \leq s\}$$

$$g_e(p) = \begin{cases} \min\{q \mid q \leq p \text{ and } \Phi_e(q) = p\}, \text{ if such a} \\ \quad q \text{ exists} \\ p, \text{ otherwise} \end{cases}$$

$$B_e = \{b_e(n)\}$$

$$B_{e,s} = \{b_e(n,s) \mid n \leq s\}$$

$$A_e = \overline{B}_e.$$

We note that while it is clear that $b_e(n) \leq b_e(n+1)$, in general b_e is not strictly increasing. Indeed, $b_e(n) < b_e(n+1)$ if and only if $\Phi_e(n+1) > b_e(n)$ and $\phi_e(n+1)\downarrow$. Also, it is clear that $b_e(n,s)$ and $g_e(p)$ are total recursive functions.

Lemma 3.1: A_e is recursively enumerable.

Proof: Simply replace $g°$ by g_e everywhere in the enumeration procedure given in §1 for $A°$. ⬢

Lemma 3.2: $\deg W_e = \deg A_e$.

Proof: This follows immediately from the equations

$$p \in W_e \iff \Phi_e(p) \leq b_e(p)$$

and

$$b_e(n) = \max\{\Phi_e(p) \mid p \leq n \text{ and } p \in W_e\}. ⬢$$

Lemma 3.3: If $\deg W_e > 0$ then

$$(\forall i)[\phi_i \text{ total } \Rightarrow (\overset{\infty}{\exists} n)[\phi_i(b_e(n)) < b_e(n+1)]].$$

Proof: Assume to the contrary that ϕ_i is total, W_e is not recursive and that for some $n_\circ$

$$(\forall n > n_\circ)[\phi_i(b_e(n)) \geq b_e(n+1)].$$

Then clearly

$$(\forall n > n_\circ)[\, b_e(n) \leq (\underbrace{\phi_i \circ \cdots \circ \phi_i}_{n})(b_e(n_\circ))],$$

from which it follows that B_e is recursive, which by virtue of Lemma 3.2 contradicts the premise that W_e is not recursive. ⬢

We also have an analog of Lemma 1.3,

Lemma 3.4: (a) $(\forall n)[B_e | b_e(n) = B_{e,b_e(n)}]$

(b) $(\forall s)[B_e | s = B_{e,s} \iff s \in B_e]$

The property of B_e described in Lemma 3.3 plays a role analagous to that of $B°$ in Lemma 1.2(a). However, since b_e dominates a given total ϕ_i only infinitely often, the construction described below attempts to find witnesses for both positive and negative requirements one at a time according to their priority by searching through the members of B_e and by examining the distance between successive members. The priorities of the positive and negative requirements are interleaved, i.e.,

$$N_1 < P_1 < N_2 < P_2 < \dots \; .$$

We describe first the enumeration procedure for the set A. As before $A_s(\overline{A}_s)$ denotes those integers $\leq s$ which have (have not) been put into A by stage s. We also denote $\overline{A}$ and $\overline{A}_s$ by B and B_s, respectively. We define the following program transformations:

$$\phi_{\tau(i)}(p) = \min\{s \mid s > p \text{ and } (\forall q \leq p)[\Phi_i^{B_s}(q) \leq s]\}$$

$$\phi_{\sigma(i)}(p) = \max\{\Phi_i(q) \mid q \leq p\}$$

$$\rho(i) = \begin{cases} \tau(j), \text{ if } i \text{ is odd and } j = (i+1)/2 \\ \sigma(j), \text{ if } i \text{ is even and } j = i/2 \\ 0, \text{ if } i = 0 \end{cases}$$

The enumeration procedure for A is similar to that given in the preceding section. As before there are two types of markers: real markers $\boxed{n}$ and phantom markers $\textcircled{m}$. However, here the real marker $\boxed{n}$ will be adjacent to the integer $b_e(n,s)$ at stage s and its final resting place will be $b_e(n)$. As suggested before we will not differentiate between active and inactive markers and will remove phantom markers instead of deactivating them. In addition, we will use two functions α_s and β_s. If $\alpha_s(n) = j$, then we are hypothesizing (i.e. hoping) at stage s that eventually $\phi_{\rho(j)}(b_e(n)) < b_e(n+1)$ and so $b_e(n)$ is a candidate witness for $N_{(j+1)/2}$ if j is odd, and for $P_{j/2}$ if j is even. If $\beta_s(j) = n$ then we are hypothesizing at stage s that $\phi_{\rho(j)}(b_e(n))\uparrow$ and again that $b_e(n)$ is a candidate witness. On this point we can regard the construction in the preceding section as having functions α_s and β_s such that $\alpha_s(n) = \beta_s(n) = n$ for all s and n. We will initially set $\alpha_1(1) = \beta_1(1) = 1$, and now proceed to describe stage s of the enumeration procedure.

Stage s:

(1) (a) Move real marker $\boxed{s}$ next to integer s

(b) Initialize α_s and β_s as follows:

$$\alpha_s(n) = \alpha_{s-1}(n), \quad \text{for } 1 \leq n < s$$

$$\beta_s(j) = \beta_{s-1}(j), \quad \text{for } 1 \leq j < s$$

$$\alpha_s(s) = \alpha_s(s-1)+1 \quad \text{and } \beta_s(s) = 0.$$

(2) Let $m = g_e(s)$, $j = \alpha_s(m)$, and $n = \beta_s(j)$.

(a) Remove phantom markers $\textcircled{m}$, $\textcircled{m+1}$, ..., $\textcircled{s}$.

(b) Move real markers $\boxed{m}$, $\boxed{m+1}$, ..., $\boxed{s}$ next to s.

(c) If $\phi_{\rho(\alpha_s(m-1))}(b_e(m-1,s)) < s$ and $\alpha_s(m-1)$ is even, then remove any phantom marker adjacent to any integer p for which $b_e(m-1,s) < p < s$.

(d) Update α_s and β_s as follows: for each $m \leq k \leq s$

$$\alpha_s(k) = \begin{cases} j+1, & \text{if } \alpha_s(m-1) = j \text{ or } g_e(b_e(m,s-1)) \neq m \\ j, & \text{otherwise} \end{cases}$$

$$\beta_s(j) = 0, \text{ if } \Phi_{\tau(j)}(b_e(m-1,s)) < s$$

$$\beta_s(j+1) = \begin{cases} m, & \text{if } \alpha_s(m) = j+1 \\ 0, & \text{otherwise} \end{cases}$$

and for each $j+1 < k \leq s$ set $\beta_s(k) = 0$.

(e) $B_s = \overline{A}_s$ now consists of all integers $\leq s$ which are adjacent to some real or phantom marker.

(3) Let j be the least integer such that $\phi_{\rho(j)}(b_e(\beta_s(j),s)) = s$. If no such j exists, then go to stage s+1. Otherwise let $n = \beta_s(j)$.

(a) If j is odd, then place a phantom marker $\textcircled{n}$ next to each member p of B_s for which $b_e(n,s) < p \leq s$.

(b) Update α_s and β_s as follows:

$$\alpha_s(k) = j, \text{ for } n < k \leq s$$

$$\beta_s(j) = \min\{k \mid b_e(n,s) < b_e(k,s)\}. \text{ ⬡}$$

Using arguments similar to those used for Lemma 2.5 and Lemma 2.6(a) we have

Lemma 3.5: (a) A is recursively enumerable.

(b) $(\forall n)(\forall t \geq b_e(n))[B_t | b_e(n) = B_{b_e(n)}]$.

(c) $(\forall n)[B | b_e(n) = B_{b_e(n)}]$.

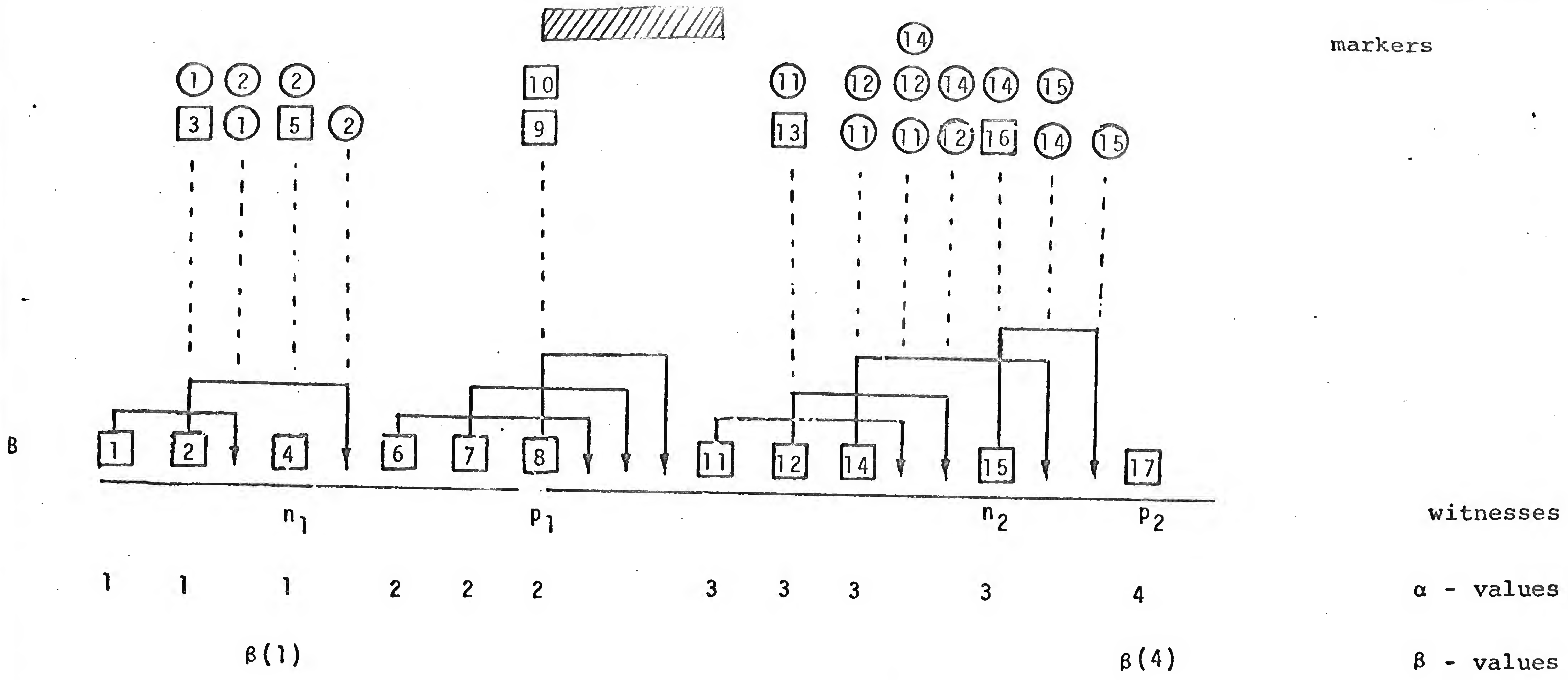
The set B has a structure which is similar to that of the set constructed in §2. Figure 2 below depicts the structure of B. One aspect of the construction which is of major importance concerns the points at which action is taken to satisfy the negative and positive requirements. From Lemma 3.3 we know that

$$\phi_i \text{ total } \Rightarrow (\overset{\infty}{\exists} n)[\phi_{\sigma(i)}(b_e(n)) < b_e(n+1)]$$

$$\phi_{\tau(i)} \text{ total } \Rightarrow (\overset{\infty}{\exists} n)[\phi_{\tau(i)}(b_e(n)) < b_e(n+1)].$$

For the negative requirement (N_i) action must be taken immediately when it is discovered that $\phi_{\tau(i)}(b_e(n)) \downarrow$, i.e, at stage $s = \phi_{\tau(i)}(b_e(n))$, even before it is known whether $\phi_{\tau(i)}(b_e(n)) < b_e(n$

Such immediate action is required to preserve the computation $\phi_{\tau(i)}(b_e(n))$. On the other hand, for the positive requirement (P_i) action is taken only after it has been confirmed that $\phi_{\sigma(i)}(b_e(n)) < b_e(n+1)$. The action taken, of course, insures that $\nu_B(b_e(n)) = \nu_{B_e}(b_e(n)) = b_e(n+1)$. In fact it is crucial that action be so delayed in order to insure that $B_e \subseteq B$, which in turn is crucial to the proof of parts (b) and (c) of Lemma 3.5.



Legend: (1) Only the outermost level of B is depicted. If $\underset{s}{\boxed{i}}\;\underset{t}{\downarrow}$ then a similar structure may occur between s and t, and so on to an arbitrary depth of nesting.

(2) ▨ indicates that no markers may occur in that interval.

FIGURE 2

Let $t_0 = 0$ and define the following:

$$n_i = \min\{p \mid p \in B_e \text{ and } p > t_{i-1} \text{ and } [\phi_{\tau(i)}(p) \uparrow \text{ or } \phi_{\tau(i)}(p) < \nu_{B_e}(p)]\}$$

$$s_i = \begin{cases} \phi_{\tau(i)}(n_i), \text{ if } \phi_{\tau(i)}(n_i)\downarrow \\ \max\{\phi_{\tau(i)}(p) \mid p \in B_e \text{ and } t_{i-1} < p < n_i\}, \\ \quad \text{otherwise.} \end{cases}$$

$$p_i = \min\{q \mid q \in B_e \text{ and } q > s_i \text{ and } [\phi_{\sigma(i)}(q)\uparrow \text{ or } \phi_{\sigma(i)}(q) < \nu_{B_e}(q)]\}$$

$$t_i = \begin{cases} \phi_{\sigma(i)}(p_i), \text{ if } \phi_{\sigma(i)}(p_i)\downarrow \\ \max\{\phi_{\sigma(i)}(q) \mid q \in B_e \text{ and } s_i < q < p_i\}, \\ \quad \text{otherwise} \end{cases}$$

Using Lemma 3.3 one can prove by induction that $(\forall n)[\lim_{s\to\infty} \alpha_s(n) \text{ exists}]$ and $(\forall j)[\lim_{s\to\infty} \beta_s(j) \text{ exists}]$.

Therefore we define

$$\alpha(n) = \lim_{s\to\infty} \alpha_s(n)$$

$$\beta(j) = \lim_{s\to\infty} \beta_s(j).$$

Let $w_j = \begin{cases} n_i, \text{ if } j \text{ is odd and } i = (j+1)/2 \\ p_i, \text{ if } j \text{ is even and } i = j/2. \end{cases}$

A careful analysis of the enumeration procedure yields, *Lemma 3.6:*

$$\text{(a)}\quad (\forall j)[\beta(j) = 0 \Longleftrightarrow \phi_{\rho(j)}(w_j)\downarrow \Longleftrightarrow w_j = \max\{p \mid \alpha(g_e(p)) = j\}.$$

$$\text{(b)}\quad (\forall j)[\beta(j) \neq 0 \Longleftrightarrow \phi_{\rho(j)}(w_j)\uparrow \Longleftrightarrow w_j = b_e(\beta(j))].$$

Then using Lemma 3.6 and Lemma 3.5(c) in a manner analogous to the proof of Lemma 2.3 we have,

Lemma 3.7:

$$\text{(a)}\quad (\forall i)[\nu_{B_e}(n_i) \neq \phi_i^B(n_i)]$$

$$\text{(b)}\quad (\forall i)[\nu_B(p_i) \neq \phi_i(p_i)]$$

From Lemma 3.5(c) it is clear that $\deg B \leq \deg B_e$. Thus, Lemma 3.7 and Lemma 3.5(a) combine to complete the proof of Theorem 3.1.

§4. Infinite Injury Priority Arguments

Theorem 3.1 is a special case of Sacks' density theorem (see [9]), which states that if C and D are recursively enumerable sets such that deg D < deg C then there exists a recursively enumerable set A such that deg D < deg A < deg C. The construction given by Sacks is an instance of what has become known as an infinite injury priority construction, since it is possible for a requirement to be injured (i.e., to be at the losing end of a conflict) infinitely many times, and hence never established. If we attempt to adapt the construction given in §3 to the density problem by replacing the relation $\Phi_i(p) \leq s$ by the relation $\Phi_i^{D_s}(p) \leq s$ we see immediately how such infinite injury can arise. The problem is that it is possible for $\phi_i^D(p)\uparrow$ and yet $(\forall s)[\phi_i^{D_s}(p)\downarrow]$. Therefore, for this modified construction, it is possible that $(\overset{\infty}{\forall} n)(\overset{\infty}{\forall} s)[\alpha_s(n) = i]$ and consequently (P_i) could interfere with the requirements (N_j), (P_j) for $j > i$. The negative and positive requirements are, of course,

$$(N_i) \quad \phi_i^B \neq \nu_C,$$

$$(P_i) \quad \phi_i^D \neq \nu_B,$$

where B is the complement of the desired set A. We now give a finite injury priority construction for the density theorem,

Theorem 4.1: (Sacks) If C and D are recursively enumerable sets such that $0 \leq \deg D < \deg C \leq 0'$, then there exists a recursively enumerable set A such that $\deg D < \deg A < \deg C$.

> The proof given below of Theorem 4.1 is incorrect. The correct construction is essentially a relativization to B_d of the construction given in Theorem 3.1 and will be given in a revised version of this paper.

joint union of A and D, $A \oplus D$, will then satisfy the conclusion of the theorem. Let C and D satisfy the hypothesis of the theorem and let $C = W_c$ and $D = W_d$. We define

$$b_+(n) = b_d(b_c(n))$$

$$b_+(n,s) = b_d(b_c(n,s),s)$$

$$g_+(p) = g_c(g_d(p))$$

$$B_+ = \{b_+(n)\}$$

$$B_{+,s} = \{b_+(n,s) \mid n \leq s\}$$

$$A_+ = \overline{B}_+ .$$

The following lemma is an easy consequence of the above definitions and the lemmas from the preceding sections.

Lemma 4.1: (a) A_+ is recursively enumerable.

(b) $\deg A_+ = \deg B_+ = \deg C$.

(c) $(\forall n)[B_d | b_+(n) = B_{d,b+(n)}]$.

(d) $(\forall i)[\phi_i^D \text{ total} \Rightarrow (\overset{\infty}{\exists} n)[\phi_i^D(b_+(n)) < b_+(n+1)]]$.

We now turn to the enumeration procedure for the desired set A. The procedure is very similar to the procedure used in the proof of Theorem 3.1. However, here α_s and β_s will deal only with the negative requirements. The positive requirements will be handled in a manner similar to that used in diagonalizations occurring in axiomatic computational complexity theory. Diagonalization in favor of (P_i) will occur at stage s if and only if (P_i) has the highest priority among all the unsatisfied requirements and it has been verified that

$$\Phi_i^{B_{d,s}}(b_+(g_+(s)-1,s)) < s.$$

A function δ_s is used to keep track of which of the positive requirements (P_j) for $j \leq s$ have been satisfied by stage s. We define α_s, β_s and δ_s initially by $\sigma_1(1) = 1 = \beta_1(1)$ and $\delta_1(1) = 0$.

Stage s:

(1) (a) Move real marker $\boxed{s}$ next to integer s.

(b) Initialize α_s, β_s, and δ_s as follows:

$$\alpha_s(n) = \alpha_{s-1}(n), \text{ for } 1 \leq n < s$$

$$\beta_s(j) = \beta_{s-1}(j), \text{ for } 1 \leq j < s$$

$$\delta_s(j) = \delta_{s-1}(j), \text{ for } 1 \leq j < s$$

$$\alpha_s(s) = \alpha_s(s-1)+1, \beta_s(s) = \delta_s(s) = 0.$$

(2) Let $m = g_+(s)$, $j = \alpha_s(m)$ and $n = \beta_s(j)$.

(a) Remove phantom markers $\textcircled{m}$, $\textcircled{m+1}$, ..., $\textcircled{s}$.

(b) Move real markers $\boxed{m}$, $\boxed{m+1}$, ..., $\boxed{s}$ next to s.

(c) For each $1 \leq j \leq s$, if $\delta_s(j) \geq m$ then set $\delta_s(j) = 0$.

(d) Let i be the least integer such that $i < \alpha_s(m-1)$ and $\delta_s(i) = 0$ and $\Phi_i^{B_{d,s}}(b_+(m-1,s)) < s$.

If such an integer i does exist then

(i) remove any phantom marker adjacent to any integer p for which $b_+(m-1,s) < p < s$,

and (ii) update α_s, β_s and δ_s as follows:

$$\delta_s(i) = m-1$$

$$\alpha_s(k) = \alpha_s(m-1), \text{ for } m \leq k \leq s$$

$$\beta_s(\alpha_s(m-1)) = m.$$

Otherwise, if no such integer exists, then update α_s, β_s and δ_s as follows:

for each $m \leq k \leq s$

$$\alpha_s(k) = \begin{cases} j+1, \text{ if } \alpha_s(m-1) = j \text{ or } g_+(b_+(m,s-1)) \neq m \\ j, \text{ otherwise} \end{cases}$$

$$\beta_s(j) = 0, \text{ if } \Phi_{\tau(j)}(b_e(m-1,s)) < s$$

$$\beta_s(j+1) = \begin{cases} m, \text{ if } \alpha_s(m) = j+1 \\ 0, \text{ otherwise} \end{cases}$$

and for each $j+1 < k \leq s$

$$\beta_s(k) = 0.$$

(e) $B_s = \overline{A}_s$ now consists of all integers $\leq s$ which are adjacent to some real or phantom marker.

(3) Let j be the least integer such that $\phi_{\tau(j)}(b_+(\beta_s(j),s)) = s$. If no such integer j exists go to stage s+1. Otherwise, let $n = \beta_s(j)$ and place a phantom marker ⓝ next to each member p of B_s for which $b_+(n,s) < p \leq s$, and update α_s and β_s as follows:

$$\alpha_s(k) = j, \text{ for } n < k \leq s$$

$$\beta_s(j) = \min\{k \mid b_+(n,s) < b_+(k,s)\}. \text{ ⬡}$$

The structure of the set B is depicted in Figure 3 below. By means similar to those used in §3 we obtain,

Lemma 4.2: (a) A is recursively enumerable.

(b) $(\forall n)[B|b_+(n) = B_{b_+(n)}]$.

(c) $(\forall n)[\alpha(n) = \lim\limits_{s\to\infty} \alpha_s(n)$ exists].

(d) $(\forall j)[\beta(j) = \lim\limits_{s\to\infty} \beta_s(j)$ exists].

Lemma 4.3: deg B < deg C.

Proof: Let

$$w_j = \begin{cases} \max\{p \mid \alpha(g_+(p)) = j\}, & \text{if } \beta(j) = 0 \\ \beta(j), & \text{if } \beta(j) \neq 0. \end{cases}$$

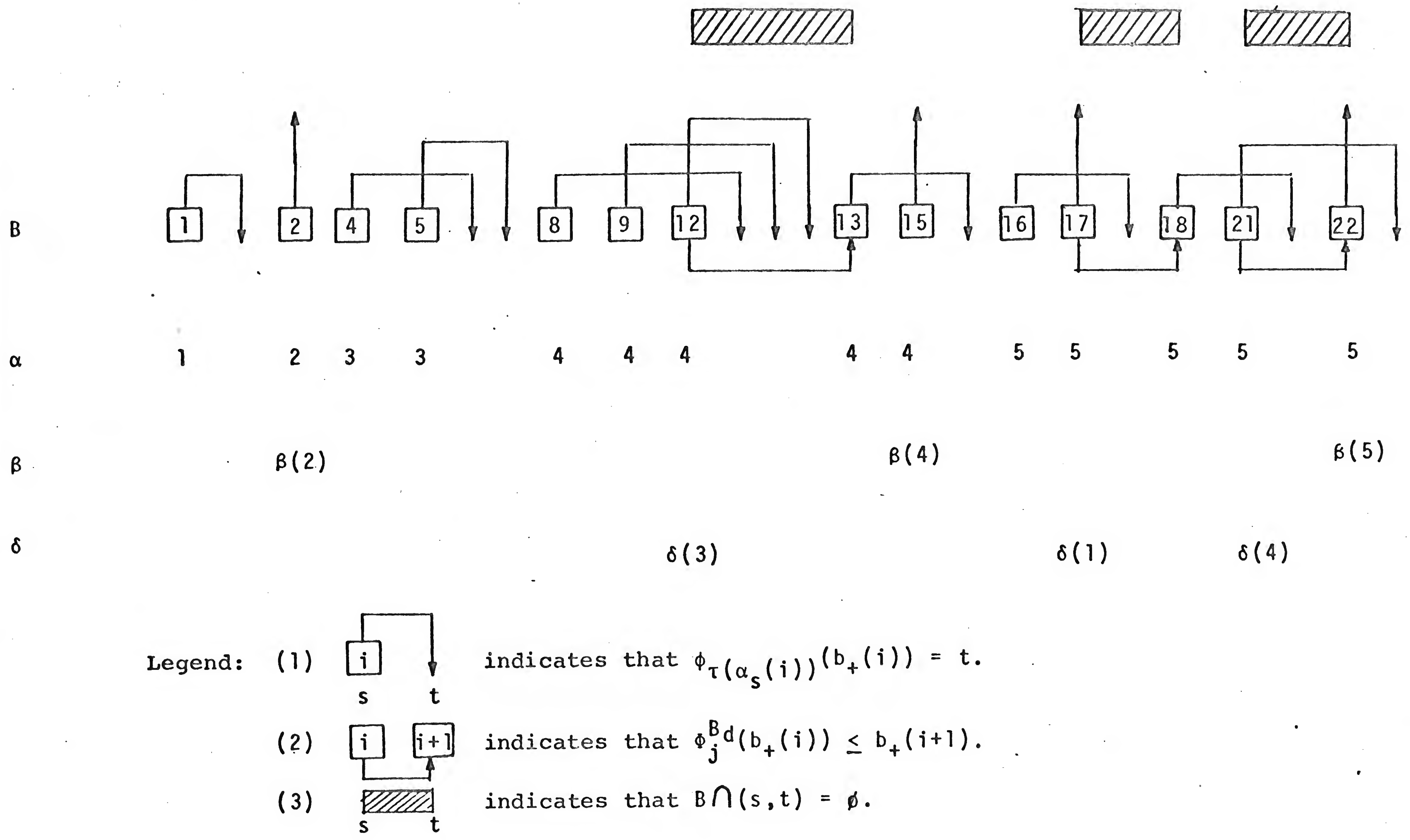
From Lemma 4.2(b) and Lemma 3.2 it suffices to show that $(\forall j)[\phi_j{}^B(w_j) \neq \nu_{B_c}(w_j)]$. From the description of the enumeration procedure one can show by an inductive argument that

$$\beta(j) = 0 \Rightarrow \phi_j{}^B(w_j) < b_+(g_+(w_j)+1)$$

$$\beta(j) \neq 0 \Rightarrow \phi_j{}^B(w_j)\uparrow. \text{ ⬡}$$

Lemma 4.4: deg D ≠ deg B.

Proof: In view of Lemma 3.2 it suffices to show that $(\forall j)[\phi_j{}^{B_d} \neq \nu_B]$. Given j, suppose that $\lim\limits_{s\to\infty} \delta_s(j)$ exists and suppose that $\delta(j) = \lim\limits_{s\to\infty} \delta_s(j) > 0$. Let $s = b_+(\delta(j)+1)$.



Legend: (1) indicates that $\phi_{\tau(\alpha_s(i))}(b_+(i)) = t$.

(2) indicates that $\phi_j^{B_d}(b_+(i)) \le b_+(i+1)$.

(3) indicates that $B \cap (s,t) = \emptyset$.

FIGURE 3

Then from the enumeration procedure we see that $\Phi_j^{B_{d,s}}(b_+(\delta(j))) < s$. By Lemma 4.1(c) we have that $B_d|s = B_{d,s}$ and hence

$$\Phi_j^{B_d|s}(b_+(\delta(j))) = \Phi_j^{B_{d,s}}(b_+(\delta(j))) < s.$$

Then from (Ω5) we obtain

$$\phi_j^{B}(b_+(\delta(j))) \leq \Phi_j(b_+(\delta(j))) < s.$$

By step (2)(d)(i) of the enumeration procedure we see that $\nu_B(b_+(\delta(j))) = s$, and hence $\phi_j^{B_d} \neq \nu_B$. Suppose now that $\lim\limits_{s\to\infty} \delta_s(j) = 0$ or does not exist. It follows from the enumeration procedure that $(\overset{\infty}{\forall} n)[\delta_{b_+(n)}(j) = 0]$ and hence from Lemma 4.1(c) that $(\overset{\infty}{\forall} n)[\Phi_j^{B_d|b_+(n+1)}(b_+(n)) \geq b_+(n+1)$ From Lemma 4.1(d) and (Ω5) and Lemma 3.2 we have that

$$(\forall i)[\phi_i^{B_d} \text{ total } \Rightarrow (\overset{\infty}{\exists} n)[\Phi_i^{B_d|b_+(n+1)}(b_+(n)) < b_+(n+1)]].$$

Therefore $\phi_j^{B_d}$ is not total and hence $\phi_j^{B_d} \neq \nu_B$. ⬣

Comparing the procedures for Theorem 3.1 and Theorem 4.1 we see an important distinction. For Theorem 3.1 the procedure can legitimately be described as a consequitive search of B_e for witnesses to (N_i) and (P_i) in order according to their priorities. What would normally be regarded as an injury in such a construction can be discounted (or justified) in this case by observing that n_i (p_i) is the first witness to (N_i) ((P_i)) after the witness to P_{i-1} ((N_i)) has been discovered. Notice that once the search for n_i or p_i has begun in earnest it is continued until it has been discovered. The case for Theorem 4.1 is different. The search for n_i even after it has begun in earnest can be interrupted (i.e., (N_i) can be injured) in order to satisfy some positive requirement with higher priority. Clearly, the construction in Theorem 4.1 is a finite injury priority construction.

We now show that it is possible to eliminate injuries in some infinite injury priority constructions. More explicitly we present a zero injury priority construction for Sacks Jump Theorem,

Theorem 4.2: (Sacks): For every set E such that E is recursively enumerable in $0'$ and $0' \leq \deg E \leq 0''$ there exists a recursively enumerable set A such that $0' < \deg A < 0'$ and $\deg A' = \deg E$.

Once again we split the proof into a series of lemmas. Let E be a set such that E is recursively enumerable in $0'$ and $0' \leq \deg E \leq 0''$. Then by a well known fact in recursive function theory (see Rogers [8]) there exists a total recursive function h such that

$$E = \{i \mid W_{h(i)} \text{ is finite}\}.$$

Define,

$$\varepsilon(n) = \max\{\Phi_{h(n)}(p) \mid \phi_{h(n)}(p)\downarrow \text{ and } p \geq 0\}$$

$$\varepsilon(n,s) = \max\{\Phi_{h(n)}(p) \mid \Phi_{h(n)}(p) \leq s \text{ and } p \leq s\}$$

$$\omega(n) = n+\max\{\varepsilon(m) \mid m \leq n \text{ and } \varepsilon(m) < \infty\}$$

$$\pi(n) = \text{cardinality}\{m \mid \varepsilon(m) < \infty\}$$

Clearly ε and ω and π are E recursive functions and

$$\varepsilon(n) < \infty \iff n \in E,$$

and $\omega(n) < \omega(n+1) < \infty$.

Next define,

$$\omega_0(n,s) = n$$

and for $0 < k \leq n$

$$\omega_k(n,s) = n+\max\{\varepsilon(i_m,s) \mid m \leq k, \text{ where } \varepsilon(i_1,s) \leq \varepsilon(i_2,s) \leq \cdots \leq \varepsilon(i_n,s)\}$$

$$\omega_k(n) = \lim_{s\to\infty} \omega_k(n,s).$$

Clearly,

$$\omega(n) = \omega_{\pi(n)}(n).$$

For $m \leq n$ define

$$\pi_{n,k}(m,s) = \text{cardinality}\{j \mid j \leq m \text{ and } \varepsilon(j,s) \leq \omega_k(n,s)\}.$$

The functions $\varepsilon(n,s)$, $\omega_k(n,s)$ and $\pi_{n,k}(m,s)$ are all total recursive. Let δ be the program transformation defined by

$$\phi_{\delta(n,k)}(p) = \min\{s \mid s > p \text{ and } k = \text{cardinality}\{m \mid \varepsilon(m,s) \leq \omega_k(n,s\}\}$$

and alter the definition of μ° so that

$$(\forall n)(\forall k \leq n)(\forall p \geq n)[\mu^\circ(S(\delta(n,k),p)) \leq p].$$

From Lemma 1.2(a) it then follows that

(4.1) $(\forall n)(\forall k \leq n)(\forall p \geq n)[\phi_{\delta(n,k)}(p) \leq b^\circ(p)]$.

The set A which we construct is "two dimensional", i.e., $A \subseteq N\times N$. For any set $X \subseteq N\times N$ we define the nth row of X by,

$$X^{(n)} = \{<n,p> \mid <n,p> \in X\},$$

and $X|s = \bigcup_{n\leq s} X^{(n)}|s$

so that $X|s \subseteq \{<p,q> \mid p,q \leq s\}$.

We use B to denote the complement of A and as before we define the set A by stages and $A_s(B_s)$ will denote the stage s approximation to $A(\overline{A})$.

The enumeration procedure for A uses phantom markers $\textcircled{n|k}$ which do not move. We define a set $B_s[n|k]$ which is intermediate to B_s and $B|s$. Implicit in the definition of $B_s[n|k]$ is the assumption that $\pi(n) = k$ and $\omega_k(n,s) = \omega(n)$.

$B_s[n|k] = \{<m,p>|m,p \leq s$ and either

(i) $<m,p>$ is adjacent to phantom marker $\textcircled{q|j}$, where either

(a) $q \leq n$ and $j \leq \pi_{n,k}(q,s)$, or

(b) $n < q \leq s$, or

(ii) $\varepsilon(m,s) < p \leq s$ and either

(a) $m > n$, or

(b) $m \leq n$ and $\varepsilon(m,s) \leq \omega_k(n,s)\}$

$$\phi_{\tau(n,k)}(p) = \min\{s|s > p \text{ and } \Phi_n^{B_s[n|k]}(p) \leq s\}$$

The placement of the phantom markers $\textcircled{n|k}$ is for the purpose of preserving the computation $\phi_{\tau(n,k)}(b^\circ(\omega_k(n)))$ as well as insuring that $B_{b^\circ(\omega_k(n))}[n|k] = B|b^\circ(\omega_k(n))$. We now proceed with the details of stage s of the enumeration procedure.

Stage s:

(1) For each $1 \leq k \leq n \leq s$,
if $g^\circ(s) \leq \omega_k(n,s)$ or if $\omega_k(n,s) > \omega_k(n,s-1)$,
then remove all phantom markers $\textcircled{n|k}$.

(2) Define,

$A_s^{(n)} = \{<n,p>|p \leq \varepsilon(n,s)$ and p is not adjacent to any phantom marker$\}$

$B_s^{(n)} = \{<n,p>|\varepsilon(n,s) < p \leq s$ or p is adjacent to some phantom marker$\}$

(3) For each $1 \leq k \leq n \leq s$,

if $\phi_{\tau(n,k)}(b^\circ(\omega_k(n,s))) = s$ or if $g^\circ(s) = \omega_k(n,s)$,

then place a phantom marker $\enclose{circle}{n|k}$ next to each member $\langle m,p\rangle$ of $B_s[n|k]$ for each $n < m \leq s$.

In a manner similar to that of the previous sections we easily obtain,

Lemma 4.5: A is recursively enumerable.

Lemma 4.6:

$$\text{(a)}\quad (\forall n)[\varepsilon(n) < \infty \Rightarrow \{\langle n,p\rangle | p>\varepsilon(n)\} \subseteq B^{(n)}$$

$$\varepsilon(n) = \infty \Rightarrow B^{(n)} \subseteq \{\langle n,p\rangle | p \leq b^\circ(\omega(n))\}]$$

$$\text{(b)}\quad (\forall n)[B_{b^\circ(\omega(n))}[n|\pi(n)] = B|b^\circ(\omega(n))]$$

$$\text{(c)}\quad (\forall n)(\forall s)[\phi_{\tau(n,\pi(n))}(b^\circ(\omega(n))) = s \Rightarrow B|s = B_s[n|\pi(n)]]$$

Proof: (a): If $\varepsilon(n) < \infty$ then for some stage s, for any $t \geq s$ $\varepsilon(n,t) = \varepsilon(n)$ and $B_t \supseteq \{\langle n,p\rangle | \varepsilon(n) < p \leq t\}$. If $\varepsilon(n) = \infty$ and $p > b^\circ(\omega(n))$ and phantom marker $\enclose{circle}{m|k}$ is adjacent to $\langle n,p\rangle$ at some stage $s \geq p$, then from step (3) of the enumeration procedure it is clear that $k \leq m < n$. If $k \leq \pi(m) \leq \pi(n)$, then $\omega_k(m) < \omega(n)$, but $\phi_{\tau(m,k)}(b^\circ(\omega_k(m))) < b^\circ(\omega(n))$ and $b^\circ(\omega_k(m)) < b^\circ(\omega(n)$ which contradicts the assumption that phantom marker $\enclose{circle}{m|k}$ is adjacent to $\langle n,p\rangle$ at stage s. Thus $k > \pi(m)$ and $\omega_k(m) = \infty$. Let $t_0 \geq s$ be such that $\varepsilon(n,t_0) > p$. Let $m_1 = \max\{m|$ phantom marker $\enclose{circle}{m|k}$ is adjacent to $\langle n,p\rangle$ at stage $t_0\}$ and let k_1 be such that phantom marker $\enclose{circle}{m_1|k_1}$ is adjacent to $\langle n,p\rangle$ at stage t_0, and let $t_1 \geq t_0$ be such that $\omega_{k_1}(m_1,t_1) > \omega_{k_1}(m_1,t_1-1)$. Then phantom marker $\enclose{circle}{m_1|k_1}$ is removed at stage t_1 and from step (3) of the enumeration procedure and the definition of $B_t[m|k]$ we see that for no stage $t > t_1$ can a phantom marker $\enclose{circle}{m|k}$ with $m \geq m_1$ be placed next to $\langle n,p\rangle$. Iterating this argument sufficiently shows that eventually there is some stage t at which no phantom marker is adjacent to $\langle n,p\rangle$ and $p \leq \varepsilon(n,t)$, so $\langle n,p\rangle \notin B$.

(b): We first observe that from (4.1) we have $\phi_{\delta(n,\pi(n))}(\omega(n)) < b^\circ(\omega(n))$, so that for each $m \leq n$, $\pi_{n,\pi(n)}(m,b^\circ(\omega(n))) = \pi(m)$. Let $s = b^\circ(\omega(n))$. If $m > n$, then from step (3) of the enumeration procedure we see that a phantom marker $\textcircled{n|\pi(n)}$ is placed next to each $\langle m,p\rangle \in B_s[n|\pi(n)]$, and, since for any $t > s$ $\omega_{\pi(n)}(n,t) = \omega(n)$ and $g^\circ(t) > \omega_{\pi(n)}(n,t)$, we see from step (1) of the enumeration procedure that these phantom markers are never removed. Suppose $m \leq n$. If $t(m,s) \leq \omega(n)$, then $\varepsilon(m) = \varepsilon(m,s)$ so that from part (a) above for any pair $\langle m,p\rangle$ such that $\varepsilon(m,s) < p \leq s$, $\langle m,p\rangle \in B|s$. If $\langle m,p\rangle$ is adjacent to phantom marker $\textcircled{q|k}$ at stage s with $q \leq n$ and $k \leq \pi(q)$, then (since $\omega_k(q) < \omega(n)$ and so for any $t > s$, $\omega_k(q,t) = \omega_k(q)$ and $g^\circ(t) > \omega_k(q)$) we obtain by step (1) of the enumeration procedure that phantom marker $\textcircled{q|k}$ is never removed and $\langle m,p\rangle \in B$. Thus we have shown that $B_s[n|\pi(n)] \subseteq B|s$. To see that $B|s \subseteq B_s[n|\pi(n)]$ it suffices in view of Lemma 4.5

to show that for every pair $\langle m,p\rangle$ such that $\langle m,p\rangle \in B_s$ and $\langle m,p\rangle \notin B_s[n|\pi(n)]$ there is a stage $t > s$ such that $\langle m,p\rangle \notin B_t$. From the proof of part (a) above we see that if $\langle m,p\rangle$ is adjacent only to phantom markers of the form $\textcircled{q|k}$ with $q \leq n$ and $k > \pi(q)$ at some stage t at which $p > \varepsilon(m,t)$, then eventually there is a stage t at which $\langle m,p\rangle \notin B_t$. Further if $m < n$ and $\omega(n) < \varepsilon(m,s) < p \leq s$ and if $\langle m,p\rangle$ is adjacent only to such markers $\textcircled{q|k}$, then since $\varepsilon(m) = \infty$ by an argument similar to that used in part (a) above we again conclude that there is a stage t such that $\langle m,p\rangle \notin B_t$. But from the definition of B_s and $B_s[n|\pi(n)]$ we see that $\langle m,p\rangle \in B_s$ and $\langle m,p\rangle \notin B_s[n|\pi(n)]$ if and only if $\langle m,p\rangle$ is adjacent only to phantom markers $\textcircled{q|k}$ at stage s with $q \leq n$ and $k > \pi(q)$ (if any), and either (1) $m < n$ and $\omega(n) < \varepsilon(m,s) < p \leq s$ or (2) $m > n$ and $p \leq \varepsilon(m,s)$

(c): Proof is similar to part (b) ⬢

In the same manner that Lemma 2.3 was proven using parts (b) and (c) of Lemma 2.2 we can show that $\deg A \neq 0'$ where $B_s[n|\pi(n)]$ replaces B_s. This combined with Lemma 4.5 yields

Lemma 4.7: $\deg A < 0'$.

Lemma 4.8: $\deg E \leq \deg A'$.

Proof: From Lemma 4.6(a) we see that

$$n \in E \Rightarrow \varepsilon(n) < \infty \Rightarrow \lim_{p\to\infty} \chi_B(n,p) = 1$$

and

$$n \notin E \Rightarrow \varepsilon(n) = \infty \Rightarrow \lim_{p\to\infty} \chi_B(n,p) = 0.$$

Therefore by the Limit Lemma of Shoenfield [10] we have $\deg E \leq \deg B' = \deg A'$. ⬡

Lemma 4.9: $\deg A' \leq \deg E$

Proof: In a manner similar to that used in Collary 2.2 we see that, for $n = S(\gamma(i),i)$

$\phi_i^B(i)\downarrow \Leftrightarrow \phi_{\tau(n,\pi(n))}(b^\circ(\omega(n))) < b^\circ(\omega(n)+1)$. Thus

$\deg A' = \deg B' \leq \max\{0', \deg E\} = \deg E$. ⬡

The above lemmas complete the proof of theorem 4.1. Notice that there are no injuries involved in the above construction. While certain diagonalizing computations must be aborted and restarted elsewhere, these abortions are caused by the movements of $b^\circ(\omega_k(n,s),s)$ and $\omega_k(n,s)$ toward their limits $b^\circ(\omega_k(n))$ and $\omega_k(n)$. Notice also that the witness n_i to (N_i) is $b^\circ(\omega(i))$.

§5. Conclusion

In the preceding sections we have seen how various finite injury and infinite injury priority constructions can be simplified. For certain of the priority constructions (finite injury and infinite injury alike) the injuries can be eliminated altogether, and one infinite injury priority construction can be reduced to one with only finite injury. Hopefully, these results show the relevance of computational complexity to the degrees of unsolvability in particular and to recursive function theory in general (see [3] for further evidence in this case). It is expected that when the remaining priority constructions are subjected to the techniques of this paper that most of them, if not all of them, can be greatly simplified. This in turn, it is hoped, will lead to better understanding and further progress in the degrees of unsolvability.

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